How small is a Billionth?

John Gough

presents some practical approaches for conceptualising very small numbers.

Children's natural curiosity about numbers, big and small (apart from other curiosity about unusual numbers, such as 77, or 1001, or 999, and "brand" numbers such as 747—the Boeing "Jumbo" number—and special date numbers, such as 1066 or 1492, when Columbus sailed the ocean blue), can lead to exploring place-value ideas. But how can these abstract concepts be experienced more concretely? This discussion considers several models for rather small numbers.

A linear model for a billionth

We'll begin with a *Linear* model, possibly the most natural (humanly intuitive) model for increasing or decreasing numerical sizes, at least for small numbers, such as can be counted on fingers.

In the playground, mark out 100 metres. This is a valuable metre-focused activity. But we will use it, for a microscopic examination, as follows:

- If we place a good metre ruler at the start of the 100 metre length, we will see centimetres, and millimetres.
- Consider the 100 metre length. It consists of 100 metres, of course.
- Each metre consists of one thousand (1000) millimetres–that is the definition of "millimetre".
- So our 100 metres contains one hundred thousand millimetres.
- That is, one millimetre is one-hundred-thousandth of the 100-metre "Unit" length.

Are we getting close to that elusive "billionth? Not really. Although the linear model is the most natural, *Big* and *Small* numbers challenge our human perception.

Incidentally, "big", "small", and "large" are not technical terms in mathematics. For example, one-millionth, which we may reasonably think of as a small number, is larger than negative one trillion, which sounds like a large number. (In "absolute" size, one trillion certainly is big. But in non-absolute terms, the "negative" alters this utterly). Around the time students venture into more challenging fractions, and decimals, they are also likely to encounter directed (negative) numbers, and the word "small" is used rather differently for positive numbers between 1 and 0, and for negative numbers less than 0. But one-billionth is, technically, multiplicatively "smaller" than one-millionth – namely, one-thousand times (i.e., multiplicatively) smaller.

If we divide each millimetre into ten equal parts (this either requires a magnifying glass, and a sharp pencil; or an enlarged scale-diagram of a "millimetre", and a bit of imagination, and faith), we have a length that is now one-tenth of one-hundred-thousandth of the 100-metre "unit": that is, a length that is one-thousand-thousandth of the unit—namely one–millionth of the unit.

We need to repeat this iterative scale-down process of dividing the current length into 10 equal parts two more times to get to the billionth part of our initial 100-metre unit. Obviously this is physically and conceptually problematic, without a very powerful micro-scope.

Of course if our initial unit is bigger than 100 metres, we will easily end up with a sensibly visible billionth part. Let's consider a kilometre as our (new) linear unit.

1 km = 1000 m = 1000 x 1000 mm.

That is, one millimetre is one-millionth of a kilometre.

Then one-thousandth of a millimetre is one thousandth of one-millionth, or one billionth of a kilometre. Not only is the one-thousandth of a millimetre, as a billionth of a kilometre, hard to see with the naked eye, the kilometre unit itself is hardly hands-on. (See the activities for making a kilometre more readily experiential in Lovitt & Clarke 1982, and Gough 1995).

An area model for a billionth

Let's try a different model for our initial unit, using area instead of length.

In the classroom, cut, fit, and glue sheets of millimetre grid paper to make a total graph-paper area of 1 square metre. Do the mathematics:

1 metre wide by 1 metre long =

1000 millimetres wide by 1000 millimetres long = 1 million square millimetres.

Hence 1 square-millimetre is one-millionth of a square-metre.

But note that we need a row of 1000 of these square-metre areas to make an area that has 1 square-millimetre as 1 billionth of the total unit-area.

Imagine painting a 1-metre wide strip along a straight kilometre of road, and put your 1 square-millimetre in the closest-left corner of the strip—that is one-billionth, shown by area. (Again, this is hardly hands-on, but interesting. Think of you, as the 1 square-millimetre, and the 1-metre wide painted kilometre strip, as the population of China! Student task: How many litres of paint would it take to paint this imagined strip?)

Note, in passing, that we can reverse the point of view, with interesting effects. One kilometre is a model for one million millimetres. Take a piece of paper, and mark a line segment that is one millimetre long: that represents you. Now use a map to find your home, and then find another place you know that is (close to) 20 kilometres distance from your home. Think about that distance: it represents the population of Australia!

A volume model for a billionth

Let's try one more model for our initial unit, using volume, instead of area or length.

Again in the classroom, construct 1 cubic-metre.

1 metre x 1 metre x 1 metre

= 1000 millimetres x 1000 millimetres x 1000 millimetres

= 1 thousand million cubic millimetres

= 1 billion cubic millimetres.

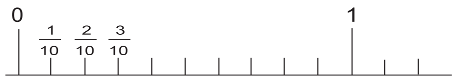
Aha! That's a bit more hands-on. One cubic-millimetre is one-billionth of a cubic-metre. Take a "hundreds-and-thousand" (a cake-decoration), as square-cornered as possible, and put this one-billionth beside your unit-cubic-metre.

Any discussion of volume as a model naturally suggests a corresponding use of mass as a model - by definition 1 cubic centimetre of water has a mass of 1 gram - so we can grasp that a "gram is a microtonne: one-millionth of a tonne. A milligram is the same as a nanotonne (10-9 of a tonne)!" (Gough, 2006 p. 139)–a billionth!

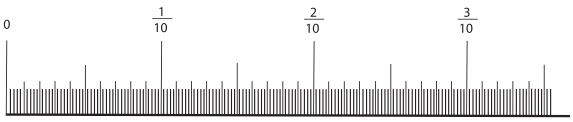
Diagrams to represent billionths

Consider some diagrams (for obvious reasons of page-size, the successive length diagrams are NOT to the same scale). First, a UNIT-LENGTH,

then dividing the Unit into Tenths

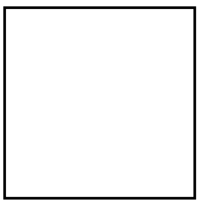


and then into Hundredths

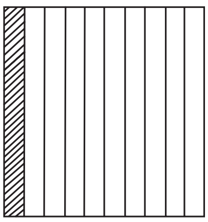


and so on

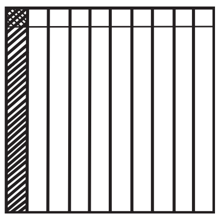
Second, staring with a Unit-Square



then dividing the unit into strips of tenths

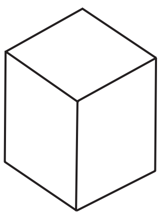


then dividing each strip into tenths of a tenth or hundredths of the original unit

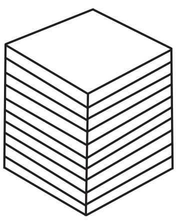


and so on,

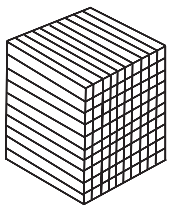
Thirdly, starting with a Unit-Cube



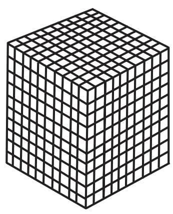
then dividing it into Tenths



and then Hundredths



and the Thousandths



and so on

(Remember, using Montessori material, or Dienes' MAB base-10 material, 1 cubic centimetre is one-thousandth of a cubic decimetre, or a litre, and 1 cubic metre contains a thousand litres, so 1 cubic metre as a model of 1 unit has a cubic centimetre as a model of one-millionth, or one-thousandth of a thousandth).

Student task: A teacher buys one sheet of millimetre graph paper, and uses this to cut out single square millimetres, and gives one to each student in a class. How many students can an A4 sheet of this graph paper supply?

Finally, make sure all these experiences of thousandths, millionths, and billionths are sensibly linked with neatly made and labelled place-value columns, and with calculator displays, both using decimal notation. What had previously been just another calculator digit a few steps to the right of the decimal-point now has a more familiar appearance, and a clearer meaning. The fundamental place-value processes of grouping into tens, and tens of tens, and so on, and of slicing into tenths, and dicing into hundredths, and so on —multiplying by 10, or, inversely, dividing by 10—has been seen in action.

This may not be literally, physically, "hands-on", but it is like being "up close and personal". Don't forget that the real focus of classroom experiences is getting strongly understood concepts inside students' brains. Mathematics can be modeled physically, and can be a lived experience, but is, eventually, and crucially, conceptually abstract, even though these abstract concepts can also be applied practically to solve real-world questions. It's a paradox of doing mathematics—in-the-head helps handle out-in-the-world,

References

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